Joint Position and Beamforming Control via Alternating Nonlinear Least-Squares with a Hierarchical Gamma Prior

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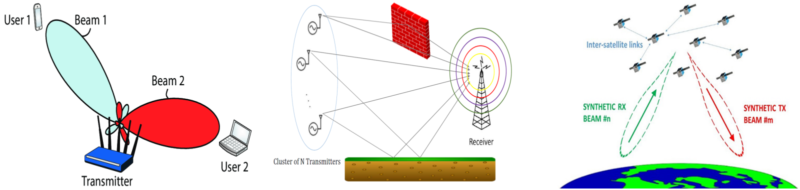
May 27, 2021

Distributed Transmit Beamforming

Transmits a signal over multiple antennas and adjusts their phase, add it constructively at the destination

→ Common in satellite communications, radars, acoustics, etc



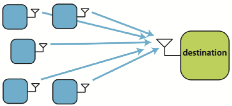
PC: Björnson E, Bengtsson M, Ottersten B. Optimal multiuser transmit beamforming: A difficult problem with a simple solution structure [lecture notes]. IEEE Signal Processing Magazine. 2014 Jun 13;31(4):142-8.

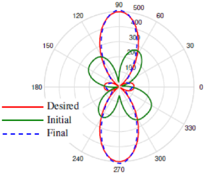
→ Distributed setting: improved security, interference reduction

⇒ Improved received signal-to-noise ratio (SNR)

Beam Synthesis: At a Glance

Consider n mobile agents at $[\mathbf{x}_i, \mathbf{y}_i]$ with omnidirectional antenna





$\Rightarrow$ $w_i \in \mathbb{C}$: excitation of node $(a_i \exp(j\alpha_i))$, θ, $\alpha \in [0, 2\pi)$

$\Rightarrow$ a_i: signal amplitude, α_i: phase, k: wave number

$\rightarrow$ Array Factor (AF) determines their ability to communicate

$$\mathrm{AF}(\theta) = \sum_{i=0}^{n-1} w_i \, e^{j(kx_i \cos(\theta) + ky_i \sin(\theta))}$$

Problem Formulation

→ Given N samples from an a priori unknown desired beam $\mathbf{AF}_d$

→ How to match desired beam at given directions using n nodes?

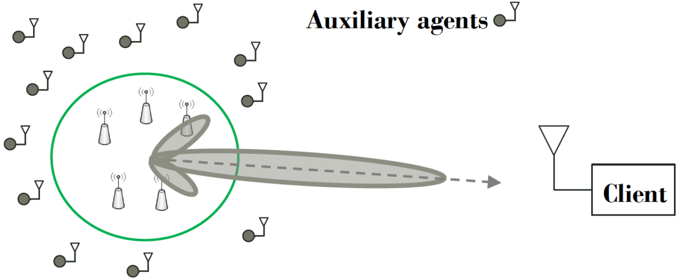
$$\min_{\{\mathbf{r_i}, w_i\}_{i=1}^{n}} \|\mathbf{AF} - \mathbf{AF}_d\|_2^2 \,,$$
$$\mathbf{AF} = \mathbf{H}(\mathbf{r})\mathbf{w}, \; ([\mathbf{H}(\mathbf{r})]_{ml} : e^{jk(x_l \cos\theta_m + y_l \sin\theta_m)})$$

⇒ Adjust position and excitation of nodes to match desired beam

→ Problem of interest:

⇒ Synthesis of array for desired pattern using set of mobile nodes

Problem Formulation



Minimize array elements filling the antenna aperture:

$\Rightarrow$ Reduced electronic beamforming cost

$\Rightarrow$ Reduction in power consumption, overall design complexity

$\rightarrow$ How to select minimum agents out of given set for desired beam ?

Agent Selection in Beamforming

Sparse beamforming: Beam synthesis with few agents as possible

$\rightarrow$ Reducing cardinality of active agent set

$\Rightarrow$ Minimizing agents with non-zero $\tilde{\mathbf{w}}$

$$\min_{\tilde{\mathbf{w}}} \|\tilde{\mathbf{w}}\|_0, \text{ s.t. } \widetilde{\mathbf{AF}}_d = \mathbf{\Phi}\tilde{\mathbf{w}}$$

$\Rightarrow$ $\widetilde{\mathbf{AF}}_d \in \mathbb{C}^{2N}$: real and imaginary part of samples

$\Rightarrow$ Beam matching in real-imaginary space is linear

$\Rightarrow$ Equivalent to constrained ℓ_0 norm minimization

Sparse Beamforming: State of Art

▪ Convex Relaxations	▪ Monte Carlo Sampling	▪ Bayesian Approach
✓ Formulates a constrained convex optimization with an equivalent convex cost function ✓ Prone to get stuck at local minima ✓ ℓ_p norm with p≈ 1 results in different global minima ✓ Sparse enough ℓ_2 norm suffers from numerous local minima	✓ Seek close-to-exact solutions ✓ Consistency requires number of samples to approach infinity ✓ Computationally complex	✓ Uses Empirical priors with flexible parameters ✓ Better mimic ℓ_0 ✓ Better route to sparsity ✓ Settle to the sparsest solution (at least in the absence of noise) and possess fewer local minima

Bayesian Approach: Overview

Gaussian likelihood model for beamforming with N samples

$$p(\widetilde{\mathbf{AF}}_d|\tilde{\mathbf{w}}) \propto (2\pi\sigma^2)^{-N/2} e^{-\frac{\|\tilde{\mathbf{AF}}_d - \mathbf{\Phi}\tilde{\mathbf{w}}\|_2^2}{2\sigma^2}}$$

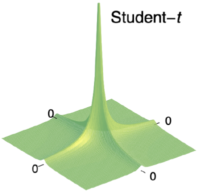
$\Rightarrow$ Error in beam matching assumed as Gaussian distribution

$\rightarrow$ Hierarchical approach: prior distribution on $\tilde{\mathbf{w}}$ and hyperparameter, γ

$\Rightarrow p(\tilde{w}_i|\gamma_i) \triangleq \mathcal{N}(0, \gamma_i),\ p(\gamma_i^{-1}) = \mathrm{Gamma}(\gamma_i^{-1}|a, b)$

$\rightarrow$ γ_i estimates controls $\tilde{w}_i$

Bayesian Approach to Agent Selection



Prior, $p(\mathbf{w})$ for conventional SBL, Gaussian-Gamma Prior [1]

→ Guarantees maximally sparse solutions[1]

⇒ Exisiting approach consider fixed layouts

⇒ Position of nodes bound reconstruction error

→ Accurate recovery with agent positions near to fictitious agents

[1] M. E. Tipping, Sparse Bayesian Learning and the Relevance Vector Machine, JMLR, 2001

Proposed Approach

$\rightarrow$ Any other choice of hierarchical prior to further shrink agent set ?

$\Rightarrow$ A sharper $\tilde{\mathbf{w}}$ distribution with a flat tail?

$\Rightarrow$ Might lead to undesirable pruning of the active agents

$\Rightarrow$ Increased error in beam matching

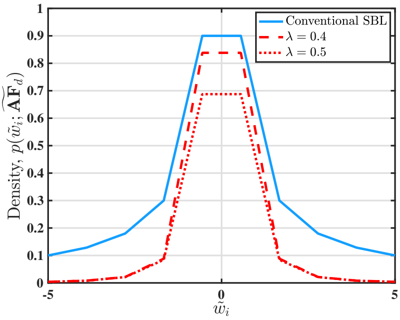
$\rightarrow$ Joint position control with modified hierarchical prior

Proposed Hierarchical Prior

Proposed hierarchical prior for sharper $\tilde{\mathbf{w}}$

$$p(\tilde{w}_i|\gamma_i) = \mathcal{N}(0, \gamma_i),\ p(\gamma_i^{-1}) = \gamma_i^{1-a} e^{-b/\gamma_i}$$

- $\rightarrow$ $\gamma_i = 0 \implies \tilde{w}_i = 0$ is null
- $\rightarrow$ Equivalent to Gamma distribution for γ_i with $a = 1, b = \frac{\lambda}{2}$



- $\rightarrow$ A sharper $\tilde{\mathbf{w}}$ distribution with a flat tail

Proposed Hierarchical Prior

→ *Marginal log-likelihood* for γ obtained by marginalizing over $\tilde{\mathbf{w}}$

$$\mathscr{L}(\gamma) = \underbrace{\boxed{\log|\Sigma_{\widetilde{\mathrm{AF}}_d}| + \widetilde{\mathrm{AF}}_d^\top \Sigma_{\widetilde{\mathrm{AF}}_d}^{-1} \widetilde{\mathrm{AF}}_d}}_{\text{Conventional SBL}} + \lambda \sum_{i}^{n} \gamma_i$$

→ Additional term prunes active set effectively

→ Expected maximization of likelihood for iterative updates of γ_i, $\tilde{w}_i$

Closed-form iterative updates for the hyperparameter γ and $\tilde{\mathbf{w}}$

$$\mu = \tilde{\mathbf{w}} = \mathbb{E}\left[\tilde{\mathbf{w}} | \widetilde{\mathrm{AF}}_d, \gamma_*\right] = \Gamma \Phi^\top \Sigma_{\widetilde{\mathrm{AF}}_d}^{-1} \widetilde{\mathrm{AF}}_d \quad (1)$$

$$\Sigma = \Gamma - \Gamma \Phi^\top \Sigma_{\widetilde{\mathrm{AF}}_d}^{-1} \Phi \Gamma \quad (2)$$

$$\gamma_i = \frac{2\left(\mu_i^2 + \Sigma_{ii}\right)}{1 + \sqrt{1 + 4\lambda(\mu_i^2 + \Sigma_{ii})}}, \text{ for all } i = 1, \ldots, n. \quad (3)$$

$\rightarrow$ Additional pruning may lead to error in beam matching

Proposed Approach: Position Control

How to reduce error from additional pruning?

→ Refined node positioning to address over pruning

→ With fixed **w**, the problem simplifies to

$$\min_{\mathbf{H}_j} \|\mathbf{AF}_d - \underbrace{\begin{bmatrix}\mathbf{H}_1 & \dots & \mathbf{H}_n\end{bmatrix}}_{\text{unknowns}, \ \mathbf{H}_j \in \mathbb{C}^N} \mathbf{w}\|_2^2$$

→ Convex set, $\mathcal{C}_1 \triangleq \left\{\mathbf{H} \in \mathbb{R}^{N \times n} \text{s.t.for all } \ j = 1, \dots, n, \mathbf{H}_j^\top \mathbf{H}_j \leq N\right\}$

→ Removes scaling ambiguity

→ Constraints **H** onto a convex set with bounded norm

Proposed Approach: Position Control

Refines agent positions using:

$\Rightarrow$ Iterative nonlinear least-squares in a constrained space

$\Rightarrow$ logarithmic transformation

$\rightarrow$ Sequential least square update of $\mathbf{H}_j$

$$\mathbf{H}_j(i+1) = \mathbf{H}_j(i) + \mathrm{diag}(\mathbf{w}(j)\mathbf{w}(j)^\top)^{-1}(\mathbf{A}\mathbf{F}_d\mathbf{w}(j)^\top - \mathbf{H}(i)\mathbf{w}\mathbf{w}(j)^\top),$$

$$\mathbf{H}_j(i+1) \leftarrow \frac{\mathbf{H}_j(i+1)}{\max(\|\mathbf{H}_j\|_2, N)}$$

$$\mathcal{C}_2 \triangleq \left\{ \mathbf{H} \in \mathbb{R}^{N\times n} \text{s.t.for all } j = 1, \ldots, n, \exists \mathbf{r}_j^*, e^{jk*\langle \mathbf{r}_j^*, (\mathbf{d}_{\theta_1} + \ldots + \mathbf{d}_{\theta_N}) \rangle} \prod_{i=1}^{N} \mathbf{H}(i,j) \right\}$$

$$\mathbf{H}_j(i+1) \leftarrow \Pi_{\mathcal{C}_2}\left[\mathbf{H}_j(i+1)\right], \text{ for all } j = 1, \ldots, n$$

Alternating Nonlinear Least-Squares with a Hierarchical Gamma Prior

→ Iteratively updates agent excitation, removes inactive agents
→ Position control of pruned agent set reduces error

Algorithm 1 Joint positioning and beamforming control

procedure REQUIRE(N samples, $\widetilde{\mathbf{AF}}_d$, set of n ($n > N+1$) agents at $\mathbf{r}$)
  Initialize : $\lambda = 1$, $\Gamma = \mathbf{I}_{n \times n}$, and $\tilde{\mathrm{w}}$
  Obtain $\mathbf{H}$ and $\mathbf{AF}$ from (4) - (5)
  while $\|\mathbf{AF}_d - \widetilde{\mathbf{AF}}\|_2 \geq \tilde{\epsilon}$ **do**
    Obtain $\boldsymbol{\Phi}$ and $\widetilde{\mathbf{AF}}$ from (7)
    while ϵ has not converged **do** ▷ sparse recovery for a given agent layout
      compute $\boldsymbol{\Sigma}$, $\tilde{\mathrm{w}}$ using (12),
      update γ_i, $\forall i = 1, \ldots, n$ using (14)
    $\tilde{\mathrm{w}} \leftarrow \mathbb{E}\left[\tilde{\mathrm{w}}|\widetilde{\mathrm{AF}}_d; \gamma_*\right] = \Gamma_* \Phi^\top \Sigma^{-1}_{\widetilde{\mathrm{AF}}_d} \widetilde{\mathrm{AF}}_d$
    Remove agents with $\tilde{\mathrm{w}} = 0$, update agent set and cardinality, n
    while $\mathbf{H}_j$ not converged **do** ▷ Position control for given sparse set
      for j=1 to n **do**
        Compute constrained least square estimate, $\mathbf{H}_j$ using (21) - (22)
        Calculate (x_j^*, y_j^*) using (23)
        Update $\mathbf{H}_j$ using (27)
    Update $\mathbf{r}$
Return: $\tilde{\mathrm{w}}$ and $\mathbf{r}$

→ Pruning agent set

→ Position Control

Numerical Experiments

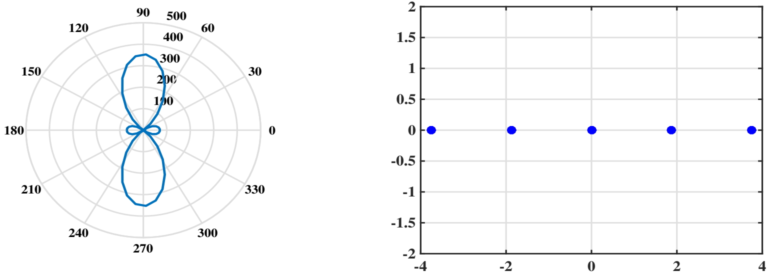


Figure: Desired Beam b) Set of fictitious agents used for desired beam

A set of 50 samples from the desired pattern by fictitious agents

- → Fictitious agents assumed to be in an equally spaced array
- → Transmits at 40 MHz with $\alpha_m = \pi/4$, a_m = 100 for all m = $1, \ldots, 5$

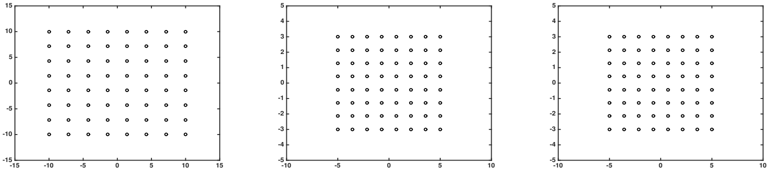
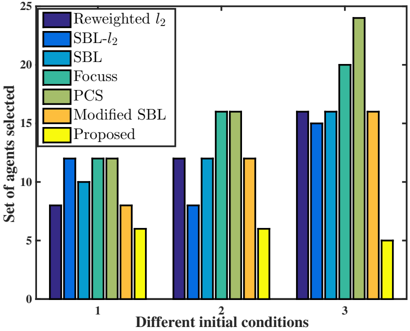
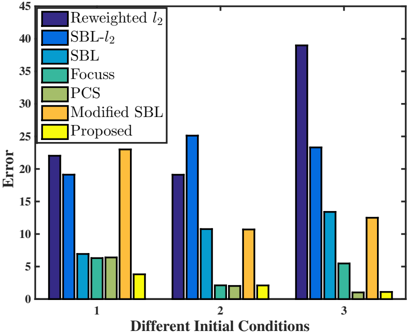


Figure: Initial agent layouts

$\rightarrow$ Different initial layouts with available set of 64 agents





Performance of Algorithm for three different initial layouts

→ Selects minimum set of active agents

→ Achieves better beam matching

Conclusion

An interweaved iterative approach for sparse beamforming

→ Adopts Bayesian framework for agent selection

→ Offers maximal sparsity for a given agent layout

→ Hierarchical prior forces high probability mass near to the null

→ Offers better shrinkage of the active agent set

→ Reduces pruning error with iterative projected block descent

→ Offers better beam matching with lower computational complexity

Thank you